

The University of Chicago

A BOUNDARY VALUE PROBLEM OF
THE CALCULUS OF VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1936

By

EVELYN PRESCOTT WIGGIN

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Reprinted from
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1. Introduction. The problem to be considered in this paper is the so-called accessory boundary value problem associated with the second variation for the problem of Bolza in the calculus of variations.

A special case of this boundary value problem, associated with a fixed end-point problem of the calculus of variations in N -space, has been discussed by Hickson (VII).^{*} He established the existence of successive characteristic numbers by the use of classical methods of the calculus of variations.

Cope (VI) defined the accessory boundary value problem for the general problem of Mayer. Morse (XI) showed that the boundary value problem associated with the problem of Bolza has infinitely many characteristic numbers, and established oscillation properties of the characteristic functions. He used methods which depend on his results for the calculus of variations in the large (VIII). Hu (XIV) studied the same problem from the standpoint of the theory of differential equations, following the procedure of Bliss (V) in establishing the existence of an infinity of characteristic numbers for a system of ordinary linear differential equations of the first order. He also deduced general oscillation and comparison theorems which are related to the theory of conjugate and focal points.

Existence theorems for characteristic numbers have also

^{*}Roman numerals refer to the papers listed in the bibliography at the end of this paper.

been established by Reid (XIII, XV) for a boundary value problem which involves the parameter in a more general form than that in which it occurs in the problems studied by Morse and Hu. His methods are related to the calculus of variations in that they use the minimizing properties of the characteristic numbers and functions, and are similar to those used by Mason (I) in treating second order differential equations. Hu's results concerning the existence of characteristic numbers are more general than those of Morse and Reid, since they are obtained for a problem of the form of the second variation which is non-singular, but for which the Clebsch condition is not necessarily satisfied. On the other hand, the problem for which oscillation and comparison theorems are established is supposed to satisfy the Clebsch condition.

E. Hölder (XXI) has recently applied the method of Lichtenstein to the study of a boundary value problem associated with a problem of Lagrange with fixed end points. His results include proofs of the existence and minimizing properties of the characteristic numbers and functions.

For the boundary value problem of this paper, the existence of infinitely many characteristic numbers is established by the use of theorems of the calculus of variations alone. The proof is made in a manner entirely analogous to that of Hickson for the simpler problem. It is carried through without the assumption that the system of differential equations satisfies any hypothesis of normality on subintervals, but only normality for the problem as a whole.

2. The Problem of Bolza. The problem of Bolza is that of finding in a class of arcs

$$(2.1) \quad y_j = y_j(x) \quad (j = 1, \dots, N; \quad x_1 \leq x \leq x_2)$$

satisfying the differential equations and end conditions

$$(2.2) \quad \phi_{\beta}(x, y, y') = 0 \quad (\beta = 1, \dots, M < N),$$

$$(2.3) \quad \psi_{\mu}[x_1, y(x_1), x_2, y(x_2)] = 0 \quad (\mu = 1, \dots, p \leq 2N + 2),$$

one which minimizes a functional of the form

$$I = G[x_1, y(x_1), x_2, y(x_2)] + \int_{x_1}^{x_2} f(x, y, y') dx.$$

The differential equations and transversality conditions which must be satisfied by a minimizing arc for this problem were deduced by Bolza (III), and necessary and sufficient conditions for a minimum have been proved by various authors.*

It will be understood that R is a region of $(2N + 1)$ - dimensional points (x, y, y') in which the functions f and ϕ_{β} have continuous partial derivatives of at least the first three orders. Similarly R' will be supposed to be a region of $(2N + 2)$ - dimensional points $(x_1, y_{j1}, x_2, y_{j2})$ in which the functions G and ψ_{μ} have continuous partial derivatives of at least the first two orders. It will also be assumed that the matrices

$$\left\| \phi_{\beta y_j'} \right\|, \quad \left\| \psi_{\mu x_1} \quad \psi_{\mu y_{j1}} \quad \psi_{\mu x_2} \quad \psi_{\mu y_{j2}} \right\|$$

have ranks M and p respectively in the regions R and R' . The numerical subscripts 1 and 2 are used to indicate the values of functions at x_1 and x_2 , respectively, and literal subscripts following the indices of functions indicate partial derivatives.

An admissible set (x, y, y') is one which is interior to R and satisfies the equations (2.2). An admissible arc $y_j = y_j(x)$ is one which is continuous and consists of a finite number of subarcs with continuously turning tangents, whose elements

*Morse (XI, XVII), Bliss (XII), Hestenes (XVI), and Reid (XVIII).

(x, y, y') are all admissible. Definitions of the equations of variation and of admissible variations given by Bliss (IX, p. 691) will be used. The tensor analysis summation convention will be followed throughout this paper.

The problem of Bolza more precisely stated is that of finding in the class of admissible arcs, satisfying the end conditions $\psi_\mu = 0$, one which minimizes the expression I.

The following properties of a minimizing arc are consequences of methods used by Bliss for problems of Lagrange and Mayer (IX, p. 692; IV, p. 309). They have been proved explicitly by Morse and Myers (X, p. 245).

THEOREM 2.1. For every minimizing arc E_{12} for the problem of Bolza there exist N constants c_j and a function

$$F = \lambda_0 f + \lambda_\beta(x) \phi_\beta(x)$$

such that the equations

$$(2.4) \quad \begin{aligned} F_{y_j'} &= \int_{x_1}^x F_{y_j} dx + c_j & (j = 1, \dots, N), \\ \phi_\beta(x, y, y') &= 0 & (\beta = 1, \dots, M) \end{aligned}$$

hold at every point of E_{12} . The first multiplier λ_0 is a constant, and the multipliers $\lambda_\beta(x)$ are continuous except possibly at the corners of E_{12} . The elements of the set $\lambda_0, \lambda_\beta(x)$ do not all vanish at any point of E_{12} . Furthermore, there exist constants g_μ ($\mu = 1, \dots, p$) such that λ_0, g_μ are not all zero, and which satisfy with the end values of E_{12} the conditions

$$(2.5) \quad \begin{aligned} \lambda_0 G_{x_1} - F(x_1) + F_{y_j'}(x_1) y_j'(x_1) + g_\mu \psi_\mu x_1 &= 0, \\ \lambda_0 G_{x_2} + F(x_2) - F_{y_j'}(x_2) y_j'(x_2) + g_\mu \psi_\mu x_2 &= 0, \\ \lambda_0 G_{y_{j1}} - F_{y_j'}(x_1) + g_\mu \psi_\mu y_{j1} &= 0, \\ \lambda_0 G_{y_{j2}} + F_{y_j'}(x_2) + g_\mu \psi_\mu y_{j2} &= 0. \end{aligned}$$

COROLLARY 2.1. At every point of E_{12} not a corner the Euler-Lagrange equations

$$(2.6) \quad \phi_{\beta}(x, y, y') = 0, \quad \frac{d}{dx} Fy_j' = Fy_j$$

must be satisfied.

COROLLARY 2.2. At every corner of E_{12} the conditions

$$(2.7) \quad Fy_j, [x, y, y'(x-0), \lambda(x-0)] = \\ Fy_j, [x, y, y'(x+0), \lambda(x+0)]$$

must be satisfied.

COROLLARY 2.3. Near every point of E_{12} which is not a corner and at which the determinant

$$(2.8) \quad \left\| \begin{array}{cc} Fy_j', y_k' & \phi_{\beta y_j'} \\ \phi_{\alpha y_k'} & 0 \cdot \delta_{\alpha\beta} \end{array} \right\| \\ (j, k = 1, \dots, N; \quad \alpha, \beta = 1, \dots, M)$$

is different from zero, the functions $y_j(x)$ defining the arc E_{12} have continuous second derivatives and the multipliers $\lambda_{\alpha}(x)$ have continuous first derivatives.

An admissible arc $y_j = y_j(x)$ with a set of multipliers $\lambda_0, \lambda_{\beta}(x)$ is called an extremal if it has continuous derivatives $y_j'(x), y_j''(x), \lambda_{\beta}'(x)$ on $x_1 x_2$, and satisfies the equations (2.6). An admissible arc is said to be non-singular if the determinant (2.8) is different from zero at every point of it. If E_{12} is a non-singular minimizing arc with no corners, it follows from Corollary 2.3 that it must be an extremal.

An admissible arc for the problem of Bolza is said to be normal if there exist p sets of admissible variations $\xi_1^{\nu}, \xi_2^{\nu}, \eta_j^{\nu}$ ($\nu = 1, \dots, p$) such that the determinant

$|\Psi_\mu(\xi^\nu, \eta^\nu)|$ is different from zero. The functions are defined by the relations

$$\begin{aligned} \Psi_\mu(\xi, \eta) = & (\psi_{\mu x_1} + \psi_{\mu y_{j_1} y_{j_1}'}) \xi_1 + \psi_{\mu y_{j_1}} \eta_{j_1} \\ & + (\psi_{\mu x_2} + \psi_{\mu y_{j_2} y_{j_2}'}) \xi_2 + \psi_{\mu y_{j_2}} \eta_{j_2}, \end{aligned}$$

where the arguments of the derivatives of ψ_μ are the end values of $E_{1,2}$. For a minimizing arc which is normal, multipliers of the form $\lambda_0 = 1$, $\lambda_\beta(x)$ exist and are unique.

In the sequel, it will be assumed that $E_{1,2}$ is a normal, non-singular extremal arc which with multipliers of the form $\lambda_0 = 1$, $\lambda_\beta(x)$, g_μ satisfies the end conditions $\psi_\mu = 0$ and the transversality conditions (2.5). Furthermore, at each element (x, y, y', λ) of $E_{1,2}$, the inequality

$$(2.9) \quad F_{y_j y_k} w_j w_k > 0$$

will be supposed to hold for every set $(w_j) \neq (0)$ which is a solution of the equations

$$\phi_{\beta y_j} w_j = 0 \quad (\beta = 1, \dots, M).$$

3. The Minimum Problem for the Second Variation of the Problem of Bolza. As was stated in Section 2, $E_{1,2}$ is supposed to be a normal extremal arc whose end values satisfy the equations $\psi_\mu = 0$. Hence, for every set of admissible variations $\xi_1, \xi_2, \eta_j(x)$ which satisfy the equations of variation $\Psi_\mu(\xi, \eta) = 0$ there exists a one parameter family of admissible arcs

$$(3.1) \quad y_j = y_j(x, b) \quad [j = 1, \dots, N; x_1(b) \leq x \leq x_2(b)]$$

whose end values satisfy the equations $\psi_\mu = 0$. The family (3.1) contains the arc $E_{1,2}$ for the parameter value $b = 0$, possesses

continuous second derivatives in b near $b = 0$, and has $\xi_1, \xi_2, \eta_j(x)$ as its variations along E_{12} (IX, p. 694). The expression I and the left members of the equations (2.2) and (2.3) are functions of b along the arcs of the family (3.1). The notations

$$\left. \frac{dx_s}{db} \right|_{b=0} = \xi_s, \quad \left. \frac{dy_{js}}{db} \right|_{b=0} = \eta_{js} \quad (s = 1, 2)$$

will be employed, and use will be made of the fact that E_{12} satisfies the equations (2.5) and (2.6) with $\lambda_0 = 1$. The value of the second derivative d^2I/db^2 for $b = 0$ may then be written in the form (VI):

$$(3.2) \quad J(\xi, \eta) = 2B[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] \\ + \int_{x_1}^{x_2} 2\omega(x, \eta, \eta') dx.$$

The form $2B$ is quadratic in the end values $\xi_1, \eta_j(x_1), \xi_2, \eta_j(x_2)$, and 2ω is the quadratic form

$$2\omega(x, \eta, \eta') = F_{y_j y_k} \eta_j \eta_k + 2F_{y_j y_k'} \eta_j \eta_k' + F_{y_j' y_k'} \eta_j' \eta_k'.$$

in which the arguments in the partial derivatives of F are those belonging to E_{12} .

The so-called accessory problem in $x\eta$ -space can now be defined. It is the problem of determining under what conditions the second variation (3.2) is non-negative in the class of sets $\xi_1, \xi_2, \eta_j(x)$ which satisfy the equations

$$(3.3) \quad \Phi_\beta(x, \eta, \eta') = \phi_{\beta y_j} \eta_j + \phi_{\beta y_j'} \eta_j' = 0 \\ (\beta = 1, \dots, M),$$

$$(3.4) \quad \Psi_\mu[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] = 0 \\ (\mu = 1, \dots, p).$$

The constants ξ_1, ξ_2 may be regarded as the initial values of functions $\eta_{N+1}(x), \eta_{N+2}(x)$ defined by the equations

$$\begin{aligned}\Phi_{N+1} &\equiv \eta_{N+1}' = 0, & \Phi_{N+2} &\equiv \eta_{N+2}' = 0, \\ \eta_{N+1}(x_1) &= \xi_1, & \eta_{N+2}(x_2) &= \xi_2.\end{aligned}$$

The subscripts 1 and α will be used to indicate the enlarged ranges of j and β , so that $i = 1, \dots, n = N + 2$; $\alpha = 1, \dots, m = M + 2$. If it is understood that the symbol η represents the vector $(\eta_1, \dots, \eta_N, \eta_{N+1}, \eta_{N+2})$, the second variation (3.2) and the equations (3.3) and (3.4) will have the following forms

$$(3.5) \quad J(\eta) = 2B[\eta(x_1), \eta(x_2)] + 2 \int_{x_1}^{x_2} \omega(x, \eta, \eta') dx,$$

$$(3.6) \quad \Phi_\alpha(x, \eta, \eta') = 0 \quad (\alpha = 1, \dots, m),$$

$$(3.7) \quad \Psi_\mu[\eta(x_1), \eta(x_2)] = 0 \quad (\mu = 1, \dots, p).$$

Since the matrix $\|\phi_{\beta\gamma j}\|$ has rank M , the matrix $\|\Phi_\alpha \eta_i\|$ ($\alpha = 1, \dots, m$; $i = 1, \dots, n$) is readily seen to have rank m .

The condition that $J(\eta)$ be non-negative in the class of arcs η which satisfy the equations (3.6) and (3.7) is equivalent to the condition that $J(\eta)$ be non-negative in the class of arcs η which satisfy (3.6), (3.7), and also the norming condition

$$(3.8) \quad \int_{x_1}^{x_2} \eta_1 \eta_1 dx = 1.$$

A minimum problem associated with the accessory problem may now be stated in symmetrical form. It is the problem of minimizing $J(\eta)$ in the class of admissible arcs η which satisfy the equations (3.6), (3.7), and (3.8).

In order to transform this problem into a problem of Bolza, the function $\eta_{n+1}(x)$ will be introduced subject to the conditions

$$(3.9) \quad \eta_{n+1}' - \eta_1 \eta_1 = 0,$$

$$(3.10) \quad \eta_{n+1}(x_1) = -1, \quad \eta_{n+1}(x_2) = 0.$$

The above minimum problem is therefore equivalent to the problem of finding the properties of a set of $n+1$ admissible functions $\eta_1(x)$, $\eta_{n+1}(x)$, which satisfies the equations (3.6), (3.7), (3.9), and (3.10), and which minimizes $J(\eta)$.

The hypothesis that the arc E_{12} is normal implies the normality of every admissible set $\eta_1(x)$, $\eta_{n+1}(x)$ for the corresponding problem defined above. For brevity, the details of the proof of this statement will be omitted.

The accessory minimum problem then satisfies all the hypotheses which were stated for the problem of Bolza in Section 2. A minimizing arc must therefore satisfy necessary conditions analogous to (2.4) and (2.5) with $\lambda_0 = 1$. Between corners this arc will define extremal segments, since E_{12} is non-singular (IX, p. 684). But an arc defined by a non-singular accessory extremal can have no corners (XVI, p. 800). In view of these results and those of Corollary 2.3 the following theorem is true.

THEOREM 3.1. For every minimizing arc T_{12} : $\eta_1 = \eta_1(x)$ ($i = 1, \dots, n$) for the accessory minimum problem, there exists a constant ρ and a function

$$\Omega(x, \eta, \eta', \mu) = \omega(x, \eta, \eta') + \mu_{\alpha(x)} \Phi_{\alpha}(x, \eta, \eta'),$$

such that the equations

$$(3.11) \quad \begin{aligned} \frac{d}{dx} \Omega \eta_1' &= \Omega \eta_1 - \rho \eta_1 & (i = 1, \dots, n), \\ \Phi_\alpha &= 0 & (\alpha = 1, \dots, m) \end{aligned}$$

hold at every point of Γ_{12} . Furthermore, there exist constants ϵ_μ ($\mu = 1, \dots, p$) which satisfy with the end values of Γ_{12} the conditions

$$(3.12) \quad \begin{aligned} B \eta_{11} - \Omega \eta_1'(x_1) + \epsilon_\mu \Psi_\mu \eta_{11} &= 0, \\ B \eta_{12} - \Omega \eta_1'(x_2) + \epsilon_\mu \Psi_\mu \eta_{12} &= 0, \\ \Psi_\mu[\eta(x_1), \eta(x_2)] &= 0 \quad (i = 1, \dots, n; \mu = 1, \dots, p). \end{aligned}$$

The functions $\eta_1(x)$, $\mu_\alpha(x)$ belonging to Γ_{12} are continuous and have continuous derivatives $\eta_1'(x)$, $\eta_1''(x)$, $\mu_\alpha'(x)$.

A value of the parameter ρ for which this boundary value problem has a continuous and non-identically vanishing solution $\eta_1(x)$, $\mu_\alpha(x)$ which has continuous derivatives $\eta_1'(x)$, $\eta_1''(x)$, $\mu_\alpha'(x)$, is said to be a characteristic number. The corresponding solution is then a characteristic solution. The normality condition which is satisfied by every admissible arc* is a necessary and sufficient condition that the boundary value problem (3.11), (3.12) have no characteristic solution $\eta_1(x)$, $\mu_\alpha(x)$ with $\eta_1(x)$ all identically zero on x_1, x_2 (XIV, p. 20).

4. Properties of the Boundary Value Problem. The class H_0 will now be defined to be the class of all admissible arcs which satisfy the equations (3.6), (3.7). The class H_0' is the

*If an accessory problem does not satisfy the condition of normality, it may always be replaced by an equivalent problem which is normal (XIX, p. 78). For a discussion of normality conditions for the general problem of Bolza the reader is referred to Bliss (XIX, pp. 46 ff.).

class of all arcs of the class H_0 which also satisfy the equation (3.8).

The following properties of the boundary value problem (3.11), (3.12) have been proved by Hn (XIV, pp. 24-37) and Morse (XI, p. 533).

LEMMA 4.1. If $\eta_1(x)$, $\mu_\alpha(x)$ is a solution of the boundary value problem (3.11), (3.12) for a given value of ρ , the expression

$$(4.1) \quad K(\eta, \rho) = J(\eta) - \rho \int_{x_1}^{x_2} \eta_1 \eta_1 \, dx$$

has the value zero. For every such solution with η belonging to the class H_0' , the expression $J(\eta)$ has the value ρ .

LEMMA 4.2. The values of the expression $J(\eta)$ in the class H_0' have a greatest lower bound ρ_0 .

LEMMA 4.3. The boundary value problem (3.11), (3.12) has only real characteristic numbers.

LEMMA 4.4. If η_1 , μ_α and η_1^* , μ_α^* are solutions of (3.11), (3.12) corresponding to distinct characteristic numbers ρ and ρ^* , then

$$\int_{x_1}^{x_2} \eta_1 \eta_1^* \, dx = 0.$$

The Euler-Lagrange equations for the expression (4.1) in the class H_0 are the equations (3.11). Since the matrix (2.8) is non-singular on $x_1 x_2$, one may introduce the canonical variables

$$(4.2) \quad \zeta_1 = \Omega_{\eta_1}(x, \eta, \eta', \mu),$$

and solve the $m + n$ equations (3.6), (4.2) for the variables η_1', μ_α .

The solutions

$$\eta_1' = \chi_1(x, \eta, \zeta), \quad \mu_\alpha = M_\alpha(x, \eta, \zeta),$$

are linear in η_1, ζ_1 . The equations (3.11) are then equivalent to

$$(4.3) \quad \begin{aligned} \eta_1' &= \chi_1(x, \eta, \zeta), \\ \zeta_1' &= \Omega \eta_1[x, \eta, \chi(x, \eta, \zeta), M(x, \eta, \zeta)] - \rho \eta_1. \end{aligned}$$

If $a_1 = a_{\theta_1}, b_1 = b_{\theta_1}$ ($\theta = 1, \dots, 2n - p$) are linearly independent solutions of

$$\Psi_{\mu \eta_1} a_1 + \Psi_{\mu \eta_2} b_1 = 0,$$

the equations (3.12) are equivalent to the $2n$ linearly independent relations

$$(4.4) \quad \begin{aligned} a_{\theta_1} [{}^B \eta_{11} - \zeta_1(x_1)] + b_{\theta_1} [{}^B \eta_{12} + \zeta_1(x_2)] &= 0, \\ \Psi_{\mu} [\eta(x_1), \eta(x_2)] &= 0, \end{aligned}$$

which are linear and homogeneous in the end values of $\eta_1(x), \zeta_1(x)$. The boundary value problem (3.11), (3.12) is then equivalent to (4.3), (4.4) in the $2n$ variables η_1, ζ_1 .

A value of ρ for which the equations (4.3), (4.4) have a continuous and non-identically vanishing solution η_1, ζ_1 , which has continuous derivatives, is a characteristic number. The corresponding solution is a characteristic solution. The normality condition implies that there is no characteristic solution of the form $\eta_1(x) \equiv 0, \zeta_1(x)$ on $x_1 x_2$.

5. The Minimum of $J(\eta)$ in the Class H_0' . If an arc $\eta_1 = \eta_1(x)$ ($i = 1, \dots, n$) minimizes $K(\eta, \rho)$ with respect to admissible arcs in the class H_0 , then it must certainly minimize

the integral

$$(5.1) \quad \int_{x_1}^{x_2} \{ 2\omega(x, \eta, \eta') - \rho \eta_1 \eta_1 \} dx$$

in the class of admissible arcs η which satisfy the differential equations $\Phi_\alpha = 0$ and join the same end-points. This minimum problem is a problem of Lagrange with fixed end-points.

The Euler-Lagrange equations of the extremals for this fixed end-point problem are the equations (4.3). The second variation of the integral (5.1) has exactly the form of the integral itself, and the accessory differential equations (IX, p. 724) are identical with the Euler-Lagrange equations of the extremals.

If the columns of the matrix

$$\left\| \begin{array}{l} \eta_{ik}(x, \rho) \\ \zeta_{ik}(x, \rho) \end{array} \right\| \quad (i = 1, \dots, n; k = 1, \dots, 2n)$$

constitute a fundamental set of solutions of the equations (4.3), the general solution of those equations has the form

$$\eta_i = \eta_{ik}(x, \rho) a_k, \quad \zeta_i = \zeta_{ik}(x, \rho) a_k,$$

where the constants a_k are arbitrary.

The equations (4.3) are said to have anormality of order $q \leq m$ on an interval $x_1 x_3$ if there exist q linearly independent accessory extremals

$$\eta_i = \eta_{i\gamma}(x, \rho), \quad \zeta_i = \zeta_{i\gamma}(x, \rho) \quad (\gamma = 1, \dots, q)$$

with $\eta_1 \equiv 0$ on $x_1 x_3$ (XVI, p. 795). It should be noted that the value of q is independent of ρ .

A value $x_3 \neq x_1$, is said to define a point 3 conjugate to 1 with respect to the equations (4.3) corresponding to a fixed

value of ρ , if there exists a solution η_1, ζ_1 of those equations whose functions $\eta_1(x)$ all vanish at x_1 and x_3 but are not identically zero on x_1x_3 . Throughout the following discussion it will be supposed that ρ has a particular fixed value. A value $x_3 \neq x_1$ determines a point 3 conjugate to 1 if and only if the matrix

$$\Delta(x_3, \rho) = \begin{vmatrix} \eta_{1k}(x_3, \rho) \\ \eta_{1k}(x_1, \rho) \end{vmatrix} \quad \begin{matrix} (i = 1, \dots, n; \\ k = 1, \dots, 2n) \end{matrix}$$

has rank less than $2n - q(x_3)$, where $q(x_3)$ is the order of anormality of the equations (4.3) on the interval x_1x_3 (XVI, p. 804).

For a particular value of ρ , the analogue of the Jacobi condition for the fixed end-point problem is that there be no point 3 conjugate to 1 such that $x_1 < x_3 \leq x_2$. This condition will be denoted by IV'. Since the problem is non-singular, this condition is sufficient to insure the existence of a family of conjugate accessory extremals with a determinant different from zero on x_1x_2 (XVII, XVIII), and hence the existence of a field of accessory extremals about an arbitrary accessory extremal. Furthermore, the Clebsch sufficient condition (2.9) for the original problem implies that the Weierstrass sufficient condition for the accessory problem holds throughout the entire field. The so-called fundamental sufficiency theorem (IX, p. 731) then implies that an accessory extremal η_1, ζ_1 which is a solution of the equations (4.3) will furnish a strong relative minimum for the integral (5.1) in the fixed end-point problem.*

If q is the order of anormality of the equations (4.3) on

*This result also follows directly from more sophisticated sufficiency theorems (XII, p. 271; XVI, p. 813; XVII, p. 157; XVIII, p. 580).

$x_1 x_2$, the first q accessory extremals in the set η_{1k}, ζ_{1k} ($k = 1, \dots, 2n$) will be so chosen that $\eta_{1\gamma} \equiv 0$ ($\gamma = 1, \dots, q$), and $\zeta_{1\gamma}$ is independent of ρ on $x_1 x_2$. If the point 2 is not conjugate to 1, the matrix

$$(5.2) \quad \left\| \begin{array}{c} \eta_{1\tau}(x_1, \rho) \\ \eta_{1\tau}(x_2, \rho) \end{array} \right\| \quad \begin{array}{l} (i = 1, \dots, n; \\ \tau = q + 1, \dots, 2n) \end{array}$$

has rank $2n - q$ (XVI, p. 804). In this case the end-points at x_1 and x_2 of every differentially admissible arc in the class H_0 may be joined by an accessory extremal (XVI, p. 809).

The functions

$$(5.3) \quad \eta_1 = \eta_{1\tau}(x, \rho) a_\tau, \quad \zeta_1 = \zeta_{1\tau}(x, \rho) a_\tau$$

define an extremal for the problem of minimizing $K(\eta, \rho)$ in the class H_0 . If the constants a_τ are so chosen that the ends of $\eta_1(x)$ join the ends of an arbitrary differentially admissible arc in the class H_0 , this extremal will satisfy the equations (3.7), and the following lemma is true.

LEMMA 5.1. If for a particular value of ρ the sufficient condition IV' is satisfied, every accessory extremal η_1, ζ_1 of the form (5.3) which satisfies the equations (3.7) will minimize the expression $K(\eta, \rho)$ with respect to admissible arcs in the class H_0 joining the points $(x_1, \eta_1(x_1))$ and $(x_2, \eta_1(x_2))$. In this case a necessary and sufficient condition that $K(\eta, \rho)$ shall be non-negative in the class H_0 is that it shall be non-negative in the class of extremals (5.3) which satisfy the equations (3.7).

When the $2n - q$ parameter family of extremals (5.3) is substituted in the expression $K(\eta, \rho)$, it becomes a quadratic

form in the variables a_τ . The expression $K(\gamma, \rho)$ and the end conditions (3.7) may be written in the form

$$(5.4) \quad K(a, \rho) = A_{\tau\pi}(\rho) a_\tau a_\pi \quad (\tau, \pi = q+1, \dots, 2n),$$

$$(5.5) \quad \Psi_\mu(a, \rho) \equiv b_{\mu\pi}(\rho) a_\pi = 0 \quad (\mu = 1, \dots, p).$$

Since the extremal E_{12} is normal, $p + q \leq 2n$, that is, $p \leq 2n - q$. Furthermore, the matrix

$$\left\| \begin{array}{cc} \Psi_\mu \eta_{j^1} & \Psi_\mu \eta_{j^2} \\ \zeta_{j\gamma}(x_1) & - \zeta_{j\gamma}(x_2) \end{array} \right\| \quad \begin{array}{l} (\mu = 1, \dots, p; \\ \gamma = 1, \dots, q; \\ j = 1, \dots, n) \end{array}$$

has rank $p + q$ (XIX, p. 53). If this matrix is multiplied on the right by the matrix (5.2), the first p rows in the product are the rows of $\|b_{\mu\pi}(\rho)\|$. All the elements in the last q rows are zero since for every differentially admissible arc η it is true that

$$\zeta_{1\gamma}(x_1) \eta_1(x_1) - \zeta_{1\gamma}(x_2) \eta_1(x_2) = 0 \quad (\gamma = 1, \dots, q)$$

(XVI, p. 800). The matrix (5.2) has rank $2n - q$ whenever x_2 is not conjugate to x_1 , and therefore the matrix $\|b_{\mu\pi}(\rho)\|$ has rank p .

It is well known from the theory of quadratic forms that the minimum value of the quadratic form (5.4) in the class of sets (a_π) which satisfy (5.5) and the equation

$$(5.6) \quad a_\pi a_\pi = 1,$$

is the smallest root of the equation

$$(5.7) \quad D(z, \rho) = \begin{vmatrix} A_{\tau\pi}(\rho) - \delta_{\tau\pi} z & b_{\tau\nu}(\rho) \\ b_{\mu\pi}(\rho) & 0 \cdot \delta_{\mu\nu} \end{vmatrix} = 0.$$

(See, for example, XX, p. 170.) In this and all following formulas the symbol δ_{jk} will be understood to have the value 1 when $j = k$, and 0 when $j \neq k$. The equation (5.7) may be written in the form

$$(5.8) \quad z^{2n-q-p} - B_1(\rho) z^{2n-q-p-1} + \dots + (-1)^{2n-q-p} p_{B_{2n-q-p}}(\rho) = 0.$$

In order that the quadratic form (5.4) shall be non-negative in the class of sets (a_π) satisfying (5.5) and (5.6), it is necessary and sufficient that the roots of the equation (5.8), which are all real (XX, p. 172), shall be positive or zero. This condition is equivalent to the existence of an r such that $0 \leq r \leq 2n - q - p$, and also

$$(5.9) \quad B_1(\rho) > 0, \quad B_2(\rho) > 0, \quad \dots, \quad B_r(\rho) > 0, \\ B_{r+1}(\rho) = 0, \quad \dots, \quad B_{2n-q-p}(\rho) = 0.$$

Then the relations (5.9) are necessary and sufficient conditions that the quadratic form (5.4) shall be non-negative in the class of sets (a_π) which satisfy (5.5). In view of Lemma 5.1, these results may be stated as follows.

LEMMA 5.2. In case the sufficient condition IV' is satisfied for a fixed value of ρ , the relations (5.9) in which the constants $B_1(\rho)$, ..., $B_{2n-q-p}(\rho)$ are defined by (5.4), (5.5), and (5.8), are necessary and sufficient conditions that $K(\eta, \rho)$ shall be non-negative in the class H_0 for the value of ρ in question.

In the first place, the sufficient condition IV' will be supposed to be satisfied for the particular value $\rho = \rho_0$, the greatest lower bound of the values of $J(\eta)$ in the class H_0' . Then the matrix $\Delta(x_3, \rho_0)$ will have rank $2n - q$ for $x_1 < x_3 \leq x_2$.

If the smallest root of the equation $D(z, \rho_0) = 0$ is positive, all the coefficients in the equation must be positive and the inequalities (5.9) must be satisfied for $\rho = \rho_0$. These coefficients are functions of ρ both explicitly and through the occurrence in them of the functions $\gamma_{i\tau}(x, \rho)$. Since they are continuous functions of ρ , there exists an $\epsilon > 0$ such that the inequalities (5.9) are satisfied for $\rho = \rho_0 + \epsilon$. Furthermore, it is possible to choose ϵ so that the rank of the matrix $\Delta(x_3, \rho_0 + \epsilon)$ is $2n - q$ also, since the order of anormality of the equations (4.3) on $x_1 x_2$ is independent of ρ . Hence there exists an $\epsilon > 0$ such that the sufficient condition IV' and the inequalities (5.9) are satisfied for $\rho = \rho_0 + \epsilon$. By Lemma 5.2, these are sufficient conditions that $K(\eta, \rho_0 + \epsilon)$ shall be non-negative in the class H_0 . But this contradicts the fact that ρ_0 is the greatest lower bound of $J(\eta)$ in the class H_0' . Therefore, in this case, the smallest root of the equation $D(z, \rho_0) = 0$ must be zero.

It follows that the coefficient $B_{2n-q-p}(\rho_0)$ must vanish. This coefficient is the determinant of the product of the matrices

$$M_0 = \begin{vmatrix} \gamma_{j\tau}(x_1, \rho_0) & \gamma_{j\tau}(x_2, \rho_0) & 0 \cdot \delta_{\tau\nu} \\ 0 \cdot \delta_{\mu j} & 0 \cdot \delta_{\mu j} & \delta_{\mu\nu} \end{vmatrix}$$

and

$$M_1 = \begin{vmatrix} B_{\gamma_{11}}[\eta_{\pi}(x_1, \rho_0), \eta_{\pi}(x_2, \rho_0)] - \zeta_{i\pi}(x_1, \rho_0) & \Psi_{\nu} \gamma_{11} \\ B_{\gamma_{12}}[\eta_{\pi}(x_1, \rho_0), \eta_{\pi}(x_2, \rho_0)] - \zeta_{i\pi}(x_1, \rho_0) & \Psi_{\nu} \gamma_{12} \\ \Psi_{\mu}[\eta_{\pi}(x_1, \rho_0), \eta_{\pi}(x_2, \rho_0)] & 0 \cdot \delta_{\mu\nu} \end{vmatrix}$$

($i, j, = 1, \dots, n$; $\tau, \pi = q+1, \dots, 2n$; $\mu, \nu = 1, \dots, p$).

The matrix $\| \Psi_\mu [\eta_\pi(x_1, \rho_0) \quad \eta_\pi(x_2, \rho_0)] \|$ is the matrix $\| b_{\mu\pi}(\rho_0) \|$ which has rank p .

Since the rank of the product of M_0 and M_1 is less than $2n - q + p$, the rows of M_0 must be orthogonal to some linear combination of the columns of M_1 . The only vectors which are orthogonal to the rows of M_0 are linear combinations of the columns of the matrix

$$M_2 = \left\| \begin{array}{c} -\zeta_{1\gamma}(x_1) \\ \zeta_{1\gamma}(x_2) \\ 0 \cdot \delta_{\mu\gamma} \end{array} \right\|$$

If a matrix M_3 of $2n + p$ rows and columns is formed by adjoining the columns of M_2 to the columns of M_1 , the columns of M_3 will be linearly dependent.

Hence there must exist $2n + p$ constants c_τ , e_μ , d_γ not all zero which satisfy the linear homogeneous equations whose coefficients are the rows of M_3 . From these equations it appears that the boundary conditions (3.12) with $\rho = \rho_0$ are satisfied by the end values of the functions

$$(5.10) \quad \begin{aligned} \eta_1 &= \eta_{1\tau}(x, \rho_0) c_\tau = \eta_{1\tau}(x, \rho_0) c_\tau + \eta_{1\gamma}(x, \rho_0) d_\gamma, \\ \zeta_1 &= \Omega_{\eta_1} = \zeta_{1\tau}(x, \rho_0) c_\tau + \zeta_{1\gamma}(x) d_\gamma. \end{aligned}$$

It is readily seen that the constants c_τ cannot all vanish, and therefore the functions (5.10) constitute a characteristic solution of the boundary value problem (4.3), (4.4) for $\rho = \rho_0$.

In the second place, if the condition IV' is not satisfied for $\rho = \rho_0$, there is a non-identically vanishing solution u_1, v_1 of the equations (4.3) for $\rho = \rho_0$ which satisfies the conditions

$$u_1(x_1) = 0 = u_1(x_3), \quad u_1(x) \neq 0 \text{ on } x_1 x_3$$

for some point x_3 such that $x_1 < x_3 \leq x_2$. In this case, the arc η defined by the equations

$$\begin{aligned}\eta_1 &= u_1(x), & x_1 &\leq x \leq x_3, \\ \eta_1 &\equiv 0, & x_3 &\leq x \leq x_2,\end{aligned}$$

will give to $K(\eta, \rho_0)$ its minimum value zero. It must therefore satisfy the equations (4.4) for $\rho = \rho_0$ with some set of functions $\zeta_1(x)$. In view of the last statement in Theorem 3.1, the functions η_1 belonging to this solution must have continuous first and second derivatives.

The boundary value problem (4.3), (4.4) will then in any case have a characteristic solution for $\rho = \rho_0$, and the following theorem is true.

THEOREM 5.1. If ρ_0 is the greatest lower bound of $J(\eta)$ in the class H_0 , there exist $r_0 > 0$ linearly independent characteristic solutions η_{1s_0} , ζ_{1s_0} ($s_0 = 1, \dots, r_0$) of the boundary value problem (4.3), (4.4) for $\rho = \rho_0$. Furthermore, these solutions may be chosen normed and orthogonal so that $J[\eta_{s_0}] = \rho_0$, and also

$$\int_{x_1}^{x_2} \eta_{1s_0} \eta_{1t_0} dt = \delta_{s_0 t_0}.$$

The following theorem is a consequence of these results, combined with those of Theorem 3.1 and Lemma 4.1.

THEOREM 5.2. A necessary and sufficient condition that $J(\eta)$ shall be non-negative in the class H_0 is that the smallest characteristic number of the boundary value problem (4.3), (4.4) is itself positive or zero.

6. Successive Characteristic Numbers and Functions. As was stated in Lemma 4.2, the expression $J(\eta)$ has a greatest lower bound ρ_0 in the class H_0' . It was proved in Section 5 that there exist r_0 linearly independent solutions $\eta_1 = \eta_{1s_0}$, $\zeta_1 = \zeta_{1s_0}$ ($s_0 = 1, \dots, r_0$) of the boundary value problem (4.3), (4.4) for $\rho = \rho_0$, and the arcs $\eta_1 = \eta_{1s_0}$ give to $K(\eta, \rho_0)$ the value zero.

The class H_1 is now defined to be the class of all arcs in the class H_0 for which the conditions

$$\int_{x_1}^{x_2} \eta_{1s_0} \eta_1 dx = 0 \quad (s_0 = 1, \dots, r_0)$$

are satisfied. The class H_1' is then defined to be the class of all arcs common to the classes H_1 and H_0' . Since H_1' is a subclass of H_0' , it follows that $J(\eta)$ has a greatest lower bound ρ_1 in H_1' , and that $\rho_1 \geq \rho_0$.

If by the method of Sections 4 and 5 it can be proved that there exist r_1 linearly independent characteristic solutions η_{1s_1} , ζ_{1s_1} ($s_1 = r_0 + 1, \dots, r_0 + r_1$) of the boundary value problem (4.3), (4.4) for $\rho = \rho_1$, and if $\rho_1 > \rho_0$, then by Lemma 4.4 it will follow that

$$\int_{x_1}^{x_2} \eta_{1s_0} \eta_{1s_1} dx = 0 \quad (s_0 = 1, \dots, r_0; \\ s_1 = r_0 + 1, \dots, r_0 + r_1).$$

The class H_2 will then be defined to be the class of all arcs of H_1 for which

$$\int_{x_1}^{x_2} \eta_{1s_1} \eta_1 dx = 0 \quad (s_1 = r_0 + 1, \dots, r_0 + r_1).$$

The class H_2' will then be made up of the normed arcs of H_2 .

Proceeding in this manner, it will be supposed that there

exist r_δ linearly independent solutions of (4.3), (4.4) for $\rho = \rho_\delta$, the greatest lower bound of $J(\eta)$ in the class H_δ' ($\delta = 0, 1, \dots, t-1$), and that $\rho_\delta > \rho_{\delta-1}$. Furthermore, these solutions are to be chosen normed and orthogonal as in Theorem 5.1. The general class H_t is then defined to be the class of admissible arcs η which satisfy the equations (3.6), (3.7), and

$$(6.1) \quad \int_{x_1}^{x_2} \eta_{i\lambda} \eta_i dx = 0 \quad (\lambda = 1, \dots, s = r_0 + r_1 + \dots + r_{t-1}).$$

The class H_t' is then made up of the normed arcs of the class H_t . Since H_t' is a subclass of H_{t-1}' , it follows that $J(\eta)$ has a greatest lower bound ρ_t in the class H_t' , and that $\rho_t \geq \rho_{t-1}$.

The proof that there exist in the class H_t a set of r_t linearly independent characteristic solutions of the boundary value problem for $\rho = \rho_t$ will be made by induction. A property of the arcs $\eta_{i\lambda}(x)$ ($\lambda = 1, \dots, s$) will be proved first.

LEMMA 6.1. The arcs $\eta_{i\lambda}(x)$ ($\lambda = 1, \dots, s$) are linearly independent on the interval $x_1 \leq x \leq x_2$.

In order to prove this lemma, suppose that s constants c_λ ($\lambda = 1, \dots, s$) satisfy the linear equation

$$c_\lambda \eta_{i\lambda} \equiv 0 \quad (i = 1, \dots, n; x_1 \leq x \leq x_2).$$

Then, in view of Lemma 4.4 and the hypothesis that the arcs in each class H_δ are chosen orthonormal,

$$0 = \int_{x_1}^{x_2} c_\lambda \eta_{i\lambda} c_\mu \eta_{i\mu} dx = c_\lambda c_\lambda.$$

Hence the multipliers c_λ are all zero.

Let $\bar{H}_t$ denote the class of arcs defined by $n + s$ differ-

entially admissible functions $\eta_1(x)$ ($i = 1, \dots, n$), $\eta_{n+\lambda}(x)$ ($\lambda = 1, \dots, s$) satisfying the differential equations

$$(6.2) \quad \begin{aligned} \eta_{n+\lambda}' &= \eta_{1\lambda} \eta_1 & (i = 1, \dots, n; \lambda = 1, \dots, s), \\ \Phi_\alpha(x, \eta, \eta') &= 0 & (\alpha = 1, \dots, m), \end{aligned}$$

and end conditions

$$(6.3) \quad \begin{aligned} \Psi_\mu[\eta(x_1), \eta(x_2)] &= 0 & (\mu = 1, \dots, p), \\ \eta_{n+\lambda}(x_1) &= 0, \quad \eta_{n+\lambda}(x_2) = 0 & (\lambda = 1, \dots, s). \end{aligned}$$

The expression $K(\eta, \rho_t)$ is non-negative in the class H_t . This condition is equivalent to the condition that $K(\eta, \rho_t) \geq 0$ in the class $\bar{H}_t$. Finally, let $\bar{H}_t'$ denote the subclass of arcs belonging to $\bar{H}_t$ and such that

$$(6.4) \quad \int_{x_1}^{x_2} \eta_h \eta_h dx = 1 \quad (h = 1, \dots, n + s).$$

We shall now consider the auxiliary problem of minimizing $K(\eta, \rho_t)$ in the class $\bar{H}_t'$.

Every admissible arc for this problem may be readily seen to be normal. The Glebsch sufficient condition is satisfied on account of (2.9). The problem therefore satisfies all the hypotheses that were required in the minimum problem discussed in Sections 3 - 5, and the results of those sections are applicable.

In the statement of results analogous to Theorem 5.1 for this case, the function Ω must be replaced by the function

$$\begin{aligned} \Gamma(x, \eta_h, \eta_h', \mu_\alpha, \sigma_\lambda, \rho_t) &= \\ \omega - (\rho_t/2) \eta_1 \eta_1' + \mu_\alpha \Phi_\alpha + \sigma_\lambda (\eta_{n+\lambda}' - \eta_{1\lambda} \eta_1) & \\ (h = 1, \dots, n + s; \quad i = 1, \dots, n; \quad \lambda = 1, \dots, s). & \end{aligned}$$

The functions ω and Φ_α involve only the variables x, η_i, η_i' .

The constant ρ_0 must be replaced by $\bar{\rho}_t$, the greatest lower bound of the values of $K(\eta, \rho_t)$ in the class $\bar{H}_t'$. Since $K(\eta, \rho_t) \geq 0$ in the class $\bar{H}_t$, it follows that $\bar{\rho}_t \geq 0$. Suppose that $\bar{\rho}_t > 0$. Since

$$K(\eta, \rho_t) - \bar{\rho}_t \int_{x_1}^{x_2} \eta_h \eta_h dx \geq 0 \quad (h = 1, \dots, n + s)$$

in the class $\bar{H}_t$, it is true that

$$K(\eta, \rho_t) - \bar{\rho}_t \int_{x_1}^{x_2} \eta_1 \eta_1 dx \geq 0 \quad (i = 1, \dots, n)$$

for all η_1 in the class H_t . The supposition that $\bar{\rho}_t > 0$ then contradicts the fact that ρ_t is the greatest lower bound of $J(\eta)$ in the class H_t' , and therefore $\bar{\rho}_t = 0$.

Then by Theorem 5.1 there surely exist $r_t > 0$ linearly independent arcs $\eta_1 = \eta_{1s_t}(x)$, $\eta_{n+\lambda} = \eta_{n+\lambda s_t}(x)$ ($s_t = s + 1, \dots, s + r_t$), each of which with corresponding multipliers $\mu_{as_t}, \sigma_{\lambda s_t}, e_{\mu s_t}$ satisfies the boundary value problem

$$\begin{aligned} \frac{d}{dx} \Gamma \eta'_h &= \Gamma \eta_h & (h = 1, \dots, n + s), \\ (6.5) \quad \eta_{n+\lambda}' &= \eta_{1\lambda} \eta_1 & (\lambda = 1, \dots, s), \\ \Phi_a(x, \eta, \eta') &= 0 & (a = 1, \dots, m), \end{aligned}$$

$$\begin{aligned} B \eta_{1^1} - \Gamma \eta_{1^1}'(x_1) + e_\mu \Psi_\mu \eta_{1^1} &= 0, \\ B \eta_{1^2} + \Gamma \eta_{1^2}'(x_2) + e_\mu \Psi_\mu \eta_{1^2} &= 0 \quad (i = 1, \dots, n), \\ (6.6) \quad \Psi_\mu [\eta(x_1), \eta(x_2)] &= 0 & (\mu = 1, \dots, p), \\ \eta_{n+\lambda}(x_1) &= 0, \quad \eta_{n+\lambda}(x_2) &= 0. \end{aligned}$$

It is a consequence of the form of the above equations that the

functions η_{is_t} ($s_t = s + 1, \dots, s + r_t$) are linearly independent.

It should be remarked that s of the equations (6.5) merely state that the multipliers σ_λ are all constants.

LEMMA 6.2. The multipliers σ_λ are all zero for every solution of the equations (6.5), (6.6).

In order to prove this lemma, use will be made of the hypothesis that each of the arcs η_λ satisfies the boundary value problem (4.3), (4.4) for some one of the values $\rho_0, \rho_1, \dots, \rho_{t-1}$ of the parameter ρ . In the equations satisfied by each of these arcs, the multipliers corresponding to σ_λ are then zero.

If $\eta_{1\delta}, \zeta_{1\delta}$ is any one of the characteristic solutions associated with $\rho = \rho_\delta$, it can be shown by a well known method of proof that

$$(\rho_t - \rho_\delta) \int_{x_1}^{x_2} \eta_1 \eta_{1\delta} dx + \sigma_\lambda \int_{x_1}^{x_2} \eta_{1\lambda} \eta_{1\delta} dx = 0$$

for every solution $\eta_1(x)$ of the equations (6.5), (6.6) (XIV, p. 27). The coefficient of $(\rho_t - \rho_\delta)$ is zero on account of (6.5), (6.6). It then follows as in the proof of Lemma 6.1 that the multipliers σ_λ are all zero.

The first n equations (6.5) with $\sigma_\lambda = 0$ are identical with the first n equations (3.11) for $\rho = \rho_t$. The first $2n + p$ equations (6.6) are the same as the boundary conditions (3.12). The arcs η_{s_t}, ζ_{s_t} ($s_t = s + 1, \dots, s + r_t$) are therefore solutions of the boundary value problem (4.3), (4.4) for $\rho = \rho_t$, and the induction is complete.

The functions $\eta_{is_t}(x)$ also satisfy the equations

$$(6.7) \quad \int_{x_1}^{x_2} \eta_{i\lambda} \eta_{is_t} dx = 0 \quad (\lambda = 1, \dots, s; \\ s_t = s + 1, \dots, s + r_t)$$

in view of (6.5) and (6.6), and are linearly independent arcs in the class H_t . Then if $\rho_t = \rho_{t-1}$, $K[\eta_{s_t}, \rho_{t-1}] = 0$, and η_{s_t} must be a linear combination of the arcs $\eta_{s_{t-1}}$ ($s_{t-1} = r_0 + \dots + r_{t-2} + 1, \dots, s$). But this is impossible in view of (6.7). Hence $\rho_t \neq \rho_{t-1}$, and since $\rho_t \geq \rho_{t-1}$, it must be true that $\rho_t > \rho_{t-1}$.

The characteristic numbers for the boundary value problem (4.3), (4.4) are roots of a permanently convergent power series in ρ , which is not identically zero since the characteristic numbers have a lower bound. Hence they are at most denumerably infinite in number.*

The results of this paper may be summarized in the following theorem.

THEOREM 6.1. The totality of the characteristic numbers and characteristic solutions of the boundary value problem (4.3), (4.4) is denumerably infinite in number, and may be represented by the symbols

$$\{\eta_{i\nu}, \zeta_{i\nu}; \lambda_\nu\} \quad (\nu = 0, 1, 2, \dots),$$

where $\lambda_t \geq \lambda_{t-1}$. These functions may furthermore be chosen normed and orthogonal in the sense that

$$\int_{x_1}^{x_2} \eta_{i\mu} \eta_{i\nu} dx = \delta_{\mu\nu}.$$

Let V_ν be the class of normed admissible arcs η satisfying the relations

$$\int_{x_1}^{x_2} \eta_{i\mu}(x) \eta_i(x) dx = 0 \quad \left(\begin{array}{l} \mu = 0, 1, \dots, \nu - 1; \\ \nu = 0, 1, 2, \dots \end{array} \right).$$

*The reader is referred to a remark of Bliss(V, p. 562).

Then λ_ν is the minimum of the expression $J(\eta)$ in the class H_ν , and moreover $J(\eta_\nu) = \lambda_\nu$.

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